From elephant poaching to the physics of a balanced (violin) bow…

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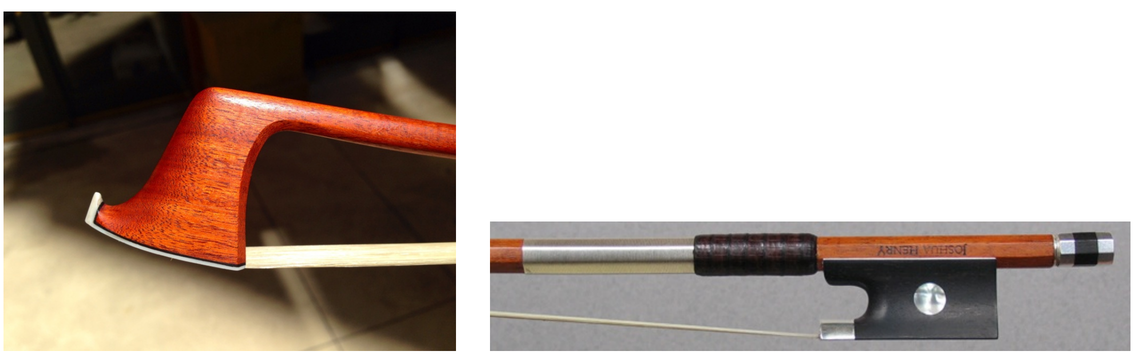
Contents

On October of 2014, prompted by the travel ban on ivory, Eric Swanson posted an article about the feasibility of swapping and ivory bow tip for a silver one. He argued that the extra weight of the silver tip would not significantly alter the bow's balance point. If one did care about the slight shift, and wanted to rebalance the bow, he pointed out that extra mass could be easily added to the winding. (If you've never given much thought to the names of bow parts, see Fig. 1.)



(a) (b)

FIG. 1: (a) Bow tip with ivory plate protecting the wood, and (b) wire winding (left) and thumb grip (center) protecting the wood from the players hand.

The question then became, can the balance point shift be easily calculated? Can the extra mass needed in the winding be easily calculated? Is the precision used by the average bow maker good enough to make the calculations, or even the rebalancing, useful? This post explains the physics behind the balance point and rebalancing, and answers these questions. If you like spoilers… the answer to the first two question is "yes", and to the last "probably not".

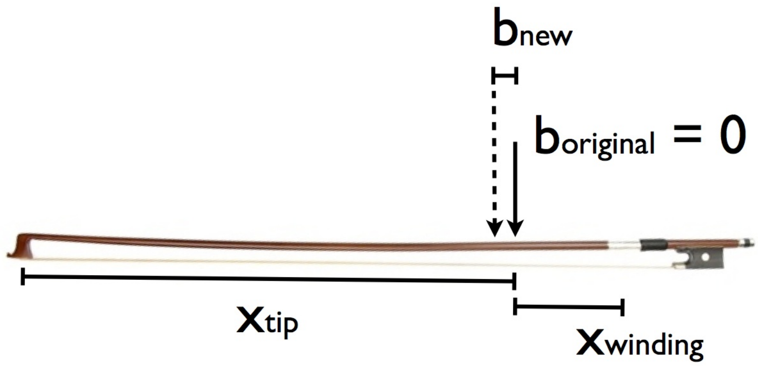


FIG. 2: Violin bow with positions labeled.

I. INTRODUCTION

To discuss the balance point of a bow, it is useful to understand it from a physics standpoint. Physicists refer to the balance point as the center of mass.[1] It is the point where, for any object of any shape, if you provide support directly under that point, the object will balance. Mathematically, instead of worrying about how gravity is pulling down on different areas of an entire object, we can imagine the force of gravity pulling down on the object at its center of mass. This equivalence allows for substantial simplification of many physics problems, including the one we have here. There are many pages on the internet that can give more details about why this is legitimate; e.g. see wikepedia's entry.

While I will do my best to use words when possible, using variables and equations often allows for more clarity. I will define variables as I go, but for convenience I list them all here.

$$\begin{aligned}
b_{\rm original} &= \text{the original balance point of the bow; this is set to be zero.} && (1)\\
b_{\rm new} &= \text{the new balance point once the tip has been changed, no adjustments made to the winding (yet).} && (2)\\
x_{\rm winding} &= \text{the distance from } b_{\rm original} \text{ to the center of mass of the winding.} && (3)\\
x_{\rm tip} &= \text{the distance from } b_{\rm original} \text{ to the center of mass of the tip piece.} && (4)\\
m_{\rm bow} &= \text{original mass of the bow, before any changes are made.} && (5)\\
m_{\rm tip} &= \text{amount of mass } \textit{added} \text{ to the tip, not the total tip mass.} && (6)\\
m_{\rm winding} &= \text{the amount of mass } \textit{added} \text{ to the winding, not the total winding mass.} && (7)
\end{aligned}$$

The distances are shown in Fig. (2).

[1] Technically, we want to use the center of gravity. The difference between center of gravity and center of mass is subtle and makes absolutely no difference in the situation here. If you are doing bow repairs or playing Mendelssohn near a black hole though. . . Well, this would be the least of your problems.

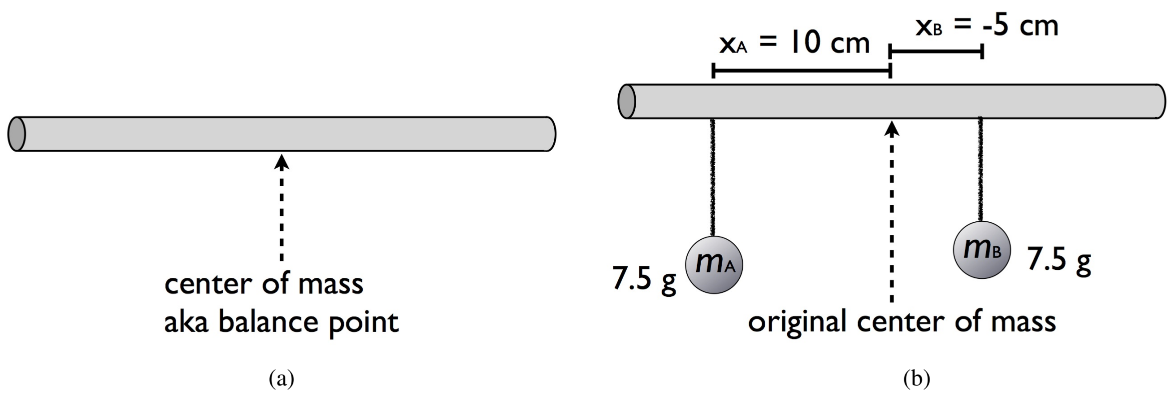


FIG. 3: (a) A stick of wood showing the center of mass, and (b) the same stick with two objects hanging from it.

II. BALANCE POINT OF A STICK WITH OBJECTS ATTACHED

Here I will explain how to calculate the balance point of a stick with a couple of objects attached to it. I will refer to this as a decorated stick. This mimics the situation we have with a bow, new tip, and possible alterations to the winding. Reading through this simpler scenario will lead to a more solid understanding of the equations and comments about (re)balancing a bow. It is not absolutely necessary for using those equations though, so feel free to skip it if you need to.

First, imagine we have a stick of uniform shape and mass, for example a dowel rod. It has a mass m_{rod} and the center of mass is in the center of this stick, as in Fig. 3 (a). To begin, we will measure all distances from this point. Later, I will show that we get the same result, but with messier math, by measuring distances from an end. I will also describe how we can easily switch to a non-uniform stick, like a bow.

Next, we will add two objects, A and B, to our stick, one to each side of its center of mass. The two objects have masses m_A and m_B and are at positions x_A and x_B as seen in Fig. 3 (b). For these calculations, it is necessary to know the position of each object, whether a mass is to the left or right of the balance point, not just its distance from that point. We will use positive and negative signs to denote distances to the left and right of the balance point, respectively. (This is opposite to the conventions one learns in school, but it is practical given that bows are held in the right hand, and, when holding a bow out flat, it points to the left.)

To calculate the center of mass of the decorated stick, we need to know the mass and position of each object: A, B, and the stick. Given this information, we use

$$x_{\text{center of mass}} = \frac{m_A x_A + m_B x_B + m_{\text{stick}} x_{\text{stick}}}{m_A + m_B + m_{\text{stick}}}, \tag{8}$$

Plugging in the numbers given in Fig. 3 (b), we have

$$\begin{aligned} x_{\text{center of mass}} &= \frac{(7.5\text{ g})(10\text{ cm}) + (7.5\text{ g})(-5\text{ cm}) + (60\text{ g})(0\text{ cm})}{7.5\text{ g} + 7.5\text{ g} + 60\text{ g}} && (9)\\ &= \frac{75 - 37.5 + 0}{75}\text{cm} && (10)\\ &= 0.5\text{ cm} && (11) \end{aligned}$$

The center of mass of this stick with two objects on it is a half of a centimeter to the left of zero, which is the center of mass of the stick alone. Note how using the center of mass of the unadorned stick means that we use a zero for x_{stick} and our equation is simplified. In fact, with the stipulation that $x_{\text{stick}} = 0$, we can write

$$x_{\text{center of mass}} = \frac{m_A x_A + m_B x_B}{m_A + m_B + m_{\text{stick}}}. \tag{12}$$

This simplified version is what we will use in the discussions about the balance point of a bow.

If you want to play with this equation more to get a better feel for it, try these variations. Put object A 5 cm to the left and object B 10 cm to the right. You will get $x_{\text{center of mass}} = -1/2$ cm; the new center of mass is a half of a centimeter to the right. With the objects in their original positions, use a mass of 15.0 g for object B. You will get $x_{\text{center of mass}} = 0$ cm; the new center of mass is in the same position as the old.

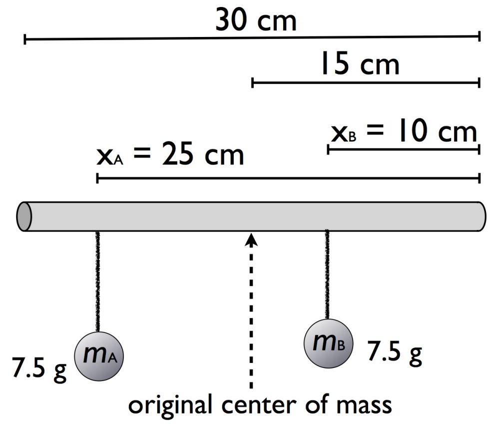


FIG. 4: Same set up as in Fig. 3 but with all measurements taken from the far right end.

What if we take all measurements from the right end of the stick? Figure 4 shows the same set up as Fig. 3 (b), but with measurements taken from the right end of the stick. In this case, we need to know the length of the stick so we can determine the distance to its center, where the center of mass is. I have given the stick a length of 30 cm, and its center of mass is at $b_{\text{stick}} = 15$ cm. Using the new numbers, we have

$$\begin{aligned} x_{\text{center of mass}} &= \frac{m_A x_A + m_B x_B + m_{\text{stick}} b_{\text{stick}}}{m_A + m_B + m_{\text{stick}}} && (13) \\ &= \frac{(7.5\text{ g})(25\text{ cm}) + (7.5\text{ g})(10\text{ cm}) + (60\text{ g})(15\text{ cm})}{7.5\text{ g} + 7.5\text{ g} + 60\text{ g}} && (14) \\ &= \frac{187.5 + 75.0 + 900}{75}\text{cm} && (15) \\ &= \frac{1162.5}{75}\text{cm} && (16) \\ &= 15.5\text{ cm}. && (17) \end{aligned}$$

The center of mass has moved a half centimeter to the left, just as before. So, setting the center of mass of the unadorned stick as the zero point, simplifies the math, and doesn't change the physical result.

What if the stick is not of uniform shape and/or mass? We can do all these calculations with a stick of unusual, asymmetric shape and mass distribution as long as we know where its center of mass is. This is very helpful when thinking about alterations of bows. The original, unaltered bow has a center of mass we can easily determine just by balancing it. You can balance it on your finger, or for better precision, a pencil or rulers edge.

III. NEW TIP, NEW BALANCE POINT

A. Location of the new balance point

We want to determine how far the center of mass of the bow moves when extra mass is added to the tip. For this, we need two sets of information. First, we need the original mass of the bow and the location of its center of mass, its balance point. Second, we need the amount of mass added to the tip and its location. This set up is shown in Fig. 5. The calculation will be simplified if we use the original balance point as our starting point, or zero, for all measurements. The tip is small, but it is not a minuscule point. To what part of the tip do we measure? – to the center of mass of the tip itself. This and other practical aspects of obtaining measurements is discussed in Sec. IV.

With the added mass of the new tip m_{tip}, and taking the center of mass of the unaltered bow to be at zero, $b_{\text{original}} = 0$, the

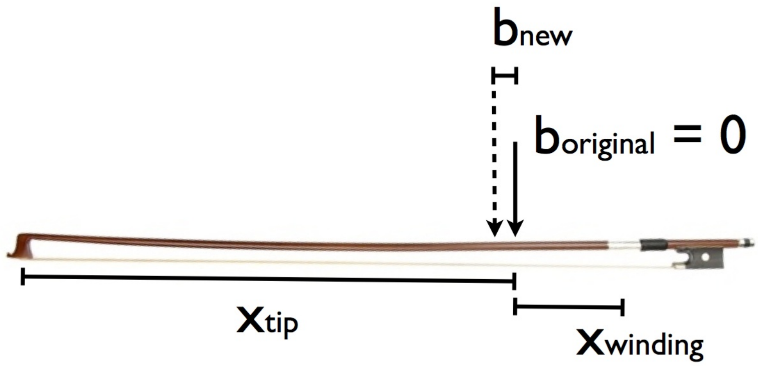


FIG. 5: Violin bow with positions labeled.

new balance point $b_{\rm new}$ is

$$b_{\rm new} = \frac{m_{\rm tip}\, x_{\rm tip} + m_{\rm bow}\, b_{\rm original}}{m_{\rm bow} + m_{\rm tip}} \tag{18}$$

$$= \frac{m_{\rm tip}}{m_{\rm bow} + m_{\rm tip}} x_{\rm tip}, \tag{19}$$

where

$$m_{\rm tip} = \text{amount of mass } \textit{added} \text{ to the tip, not the total tip mass} \tag{20}$$

$$x_{\rm tip} = \text{the distance to the center of mass of the tip piece} \tag{21}$$

$$m_{\rm bow} = \text{original mass of the bow, before any changes are made.} \tag{22}$$

B. The error in the determination

Measurements all have an error associated with them. For example, typically the bow mass is measured within a tenth of a gram. We have then $m_{\rm bow} = 60.0$ g and an error $\sigma_{m_{\rm bow}} = 0.1$ g, which can be written more compactly as $m_{\rm bow} = 60.0(1)$ g. Knowing the errors on each measured quantity, we can determine the error on our calculated value of $b_{\rm new}$. We have

$$\left(\frac{\sigma_{b_{\rm new}}}{b_{\rm new}}\right)^2 = \left(\frac{\sigma_{m_{\rm tip}}}{m_{\rm tip}}\right)^2 + \left(\frac{\sigma_{x_{\rm tip}}}{x_{\rm tip}}\right)^2 + \left(\frac{\sigma_{m_{\rm bow}} + \sigma_{m_{\rm tip}}}{m_{\rm bow} + m_{\rm tip}}\right)^2, \tag{23}$$

where the σ variable represents the error of the associated quantity.

C. Comments on precision

Now that we have the equations for the new balance point and its error, it is worth thinking about the precision of typical measurements done by a bow maker. As an example, we'll use the numbers from the post by Eric Swanson about replacing an ivory tip with a silver one. In that case, the new tip was 0.30 grams heavier than the old tip. The typical mass of a violin bow is 60.0 grams, so the mass of the bow with the new tip is 60.3 g. A reasonable distance from the (original) balance point to the center of the tip plate is 50 centimeters. Using these values we have

$$b_{\rm new} = \frac{0.30 \text{ g}}{60.3 \text{ g}} \times 50.0 \text{ cm} = 0.0050 \times 50.0 \text{ cm} = 0.25 \text{ cm}, \tag{24}$$

or 2.5 millimeters, which is less than an 1/8 of an inch.

As for precision, masses are typically measured to within a tenth of a gram. An average bow will have a mass then of $m_{\rm bow} = 60.0(1)$ g. Changing a bow tip from ivory to silver adds a mass of $m_{\rm tip} = 0.3(1)$ g. Measurements of length typically have an error of at least a millimeter. We have then $x_{\rm tip} = 50.0(1)$ cm.

$$\left(\frac{\sigma_{b_{\rm new}}}{b_{\rm new}}\right)^2 \;=\; \left(\frac{0.1}{0.3}\right)^2 + \left(\frac{0.1}{50}\right)^2 + \left(\frac{0.1+0.1}{60.0+0.3}\right)^2. \tag{25}$$

On the right-hand side, the quantities in the last two parentheses are very small compared to that in the first. We can neglect them, which gives us

$$\left(\frac{\sigma_{b_{\rm new}}}{b_{\rm new}}\right)^2 \;\approx\; \left(\frac{0.1}{0.3}\right)^2, \tag{26}$$

or simply

$$\frac{\sigma_{b_{\rm new}}}{b_{\rm new}} \;\approx\; \frac{1}{3}. \tag{27}$$

This means that the error in our calculation of $b_{\rm new}$ is about 1/3 of the value of $b_{\rm new}$ itself – quite large. Specifically, $b_{\rm new} = 2.5(8)$ mm $\approx 3(1)$ mm.

IV. REBALANCING THE BOW

Although the change to the balance point is quite small, someone might want to correct for it. Below, I consider how that could be done.

A. Comments on trial and error and typical precision

One way of correcting the bow's balance point (returning its center of mass to the original point) is simply trial and error. One needs add mass to the end opposite of the tip. A simple place to do this is in the winding. A bow maker can replace the original winding with a slightly thicker wire adding more mass. They could also extend the winding, further adding to the mass. By marking the balance point of the bow before replacing the tip, they can then simply add coils to the new winding one at a time until the balance point is restored to its original point. This trial-and-error method is very practical and easy to implement. Its limitation is that it is only as good as the bow maker's ability to measure the balance point in the first place. As discussed, this could typically have an error of a millimeter or two, comparable to the shift induced by the change in tip material

B. Two equations and their relative merits

It is possible to calculate the amount of mass needed to rebalance the bow. There are two ways to approach this, one uses the center-of-mass equation we have been discussing, another uses equations from the study of rotational equilibrium. The derivations are straightforward, but would take up more space that I'd like here, so I simply state the two results below. They depend on different inputs and so have different strengths when it comes to actual applications.

From center-of-mass equations

$$m_{\rm winding} \;=\; \frac{b_{\rm new}(m_{\rm bow} + m_{\rm tip})}{-x_{\rm winding}}, \tag{28}$$

and from rotational equilibrium

$$m_{\rm winding} = \frac{x_{\rm tip}}{x_{\rm winding}} m_{\rm tip}, \tag{29}$$

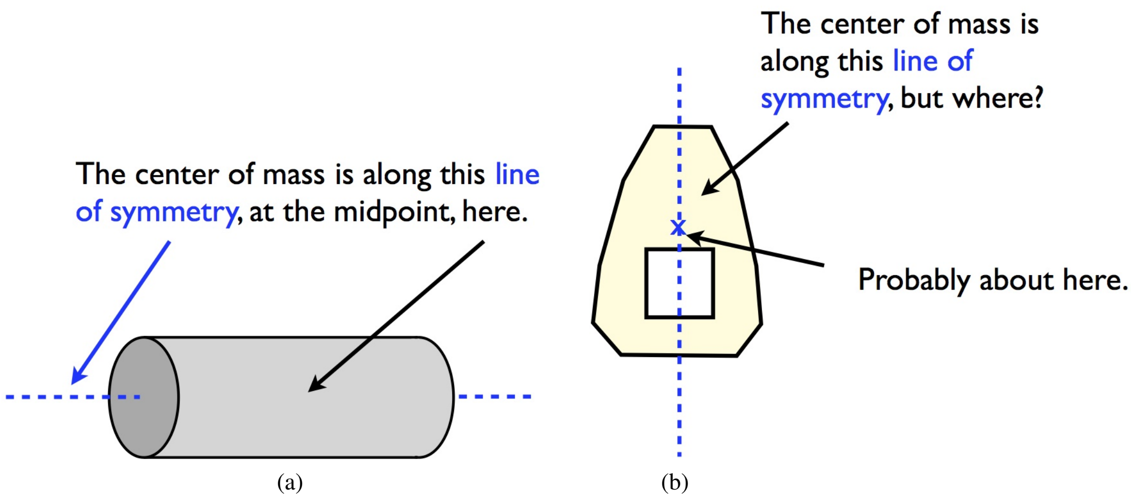


FIG. 6: Center of mass for (a) winding and (b) tip.

where

$$
\begin{aligned}
m_{\text{winding}} &= \text{is the amount of mass } \textit{added} \text{ to the winding, not the total winding mass} && (30)\\
b_{\text{new}} &= \text{the new balance point once the tip has been changed, no adjustments made to the winding (yet)} && (31)\\
m_{\text{bow}} &= \text{original mass of the bow, before any changes are made} && (32)\\
m_{\text{tip}} &= \text{amount of mass } \textit{added} \text{ to the tip, not the total tip mass} && (33)\\
x_{\text{winding}} &= \text{the distance from } b_{\text{original}} \text{ to the center of mass of the winding} && (34)\\
x_{\text{tip}} &= \text{the distance from } b_{\text{original}} \text{ to the center of mass of the tip piece} && (35)\\
& && (36)
\end{aligned}
$$

These are shown in Fig. (2). Note that x_{winding} and x_{tip} are distances to the center of mass of the winding and tip, respectively. How do we know where these are located? The winding, the coil of wire itself, is relatively simple because it is a symmetric shape – basically a hollow cylinder. Its center of mass is at the center of the hollow cylinder. We don't care about where it lies inside the bow. We just need to know where its center of mass lies along the bow stick, and that is at the midpoint of the coiled winding. This is sketched in Fig. 6 (a). Note, the winding generally extends underneath the thumb grip, so its midpoint is not the midpoint of the visible wire. Finding the center of mass of the tip is more difficult because it has an odd shape, shown in Fig. 6 (b). One could determine it exactly by using the methods described on this webpage. It is probably sufficient though to take the center of mass as being halfway along the tip, as suggested in Fig. 6 (b). That may be off by a millimeter or two, but that is the accuracy of other measurements here as well.

There will be an error on the calculation of m_{winding} in each case. For Eq. (28) it is

$$\left(\frac{\sigma_{m_{\text{winding}}}}{m_{\text{winding}}}\right)^2 = \left(\frac{\sigma_{b_{\text{new}}}}{b_{\text{new}}}\right)^2 + \left(\frac{\sigma_{m_{\text{bow}}} + \sigma_{m_{\text{tip}}}}{m_{\text{bow}} + m_{\text{tip}}}\right)^2 + \left(\frac{\sigma_{x_{\text{winding}}}}{x_{\text{winding}}}\right)^2. \tag{37}$$

Using our example numbers and errors, and assuming the center of mass of the winding is 15.0 cm from the balance point with an error of 1 mm, we have something like

$$\left(\frac{\sigma_{m_{\text{winding}}}}{m_{\text{winding}}}\right)^2 = \left(\frac{0.1}{0.3}\right)^2 + \left(\frac{0.1+0.1}{60.0+0.3}\right)^2 + \left(\frac{0.1}{15.0}\right)^2. \tag{38}$$

The term with the error from b_{new} dominates. So, we can neglect the last two terms on the right hand side and take the square root leaving,

$$\frac{\sigma_{m_{\text{winding}}}}{m_{\text{winding}}} \approx \frac{\sigma_{b_{\text{new}}}}{b_{\text{new}}}. \tag{39}$$

Using our example numbers, this has a value of 1/3, or 33%. This is quite a large error, meaning our result is not very precise.

For Eq. (29) it is

$$\left(\frac{\sigma_{m_{\rm winding}}}{m_{\rm winding}}\right)^2 = \left(\frac{\sigma_{x_{\rm tip}}}{x_{\rm tip}}\right)^2 + \left(\frac{\sigma_{x_{\rm winding}}}{x_{\rm winding}}\right)^2 + \left(\frac{\sigma_{m_{\rm tip}}}{m_{\rm tip}}\right)^2 . \tag{40}$$

and our example numbers yield

$$\left(\frac{\sigma_{m_{\rm winding}}}{m_{\rm winding}}\right)^2 = \left(\frac{0.1}{50.0}\right)^2 + \left(\frac{0.1}{15.0}\right)^2 + \left(\frac{0.1}{0.3}\right)^2 . \tag{41}$$

The term with the error from $m_{\rm tip}$ dominates here and we can write

$$\frac{\sigma_{m_{\rm winding}}}{m_{\rm winding}} \approx \frac{\sigma_{m_{\rm tip}}}{m_{\rm tip}} \tag{42}$$

which happens to again have the value of 1/3, or 33%, using our example numbers.

C. Practicality of using the equations – an example

Now that we have equations for calculating the mass needed to rebalance the tip and its error, we can discuss the merits of each equation and their ease of application (or not) in the real world. For convenience, I restate the equations here:

$$m_{\rm winding} = \frac{b_{\rm new}(m_{\rm bow} + m_{\rm tip})}{-x_{\rm winding}}, \quad \text{with error} \tag{43}$$

$$\frac{\sigma_{m_{\rm winding}}}{m_{\rm winding}} \approx \frac{\sigma_{b_{\rm new}}}{b_{\rm new}}, \tag{44}$$

or

$$m_{\rm winding} = \frac{x_{\rm tip}}{x_{\rm winding}} m_{\rm tip}, \quad \text{with error} \tag{45}$$

$$\frac{\sigma_{m_{\rm winding}}}{m_{\rm winding}} \approx \frac{\sigma_{m_{\rm tip}}}{m_{\rm tip}} \tag{46}$$

Which equation is best to use? That depends on what measurements are easiest to take. Equation 43 has the advantage of not having to determine $x_{\rm tip}$ to use it. As far as accuracy goes, if you used Eq. (43), you want to determine $b_{\rm new}$ well so that $\sigma_{b_{\rm new}}$ is small and your error on $m_{\rm winding}$ is small. If you use Eq. (45), you will want to determine $m_{\rm tip}$ wells so that $\sigma_{m_{\rm tip}}$ is small. If you happen to have a scale that measures masses to a hundredth of a gram, you'll have good precision on $m_{\rm tip}$. If not, you can try to rig something, like balancing the bow on a thin rod, to obtain good precision on $b_{\rm new}$. In either case, you need to be more accurate than the average bow maker in order to determine $m_{\rm winding}$ well and to make the change to the winding to rebalance the bow.

V. DOES IT MATTER?

Probably not. Unless you (or your customers) are sensitive to millimeter changes in the balance point, swapping out ivory for silver won't matter. If you have other materials, you can use your own measured values to determine the shift in the balance point, which is likely small as well.

If you do need to rebalance within a couple of millimeters, you will need to be able to measure either the shift in balance point to sum-millimeter precision or the added tip mass to 0.01 g precision or better. Your options are:

- Rebalance by trial and error. You will need precision in your measurement of the balance point.
- Use Eq. 43 to calculate the mass to add the winding. You will need precision in your measurement of the balance point.
- Use Eq. 45 to calculate the mass to add the winding. You will need precision in your measurement of the added tip mass.

Finally, the heavier the bow, the less the balance point will shift for a given change in tip mass. So, for a viola, cello, or bass bow, there is even less of an issue.